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Analytical approach to localization of electromagnetic waves in two-dimensional random media

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Localization of electromagnetic waves in two-dimensional random media is studied analytically. A simple but realistic model, based directly on the Maxwell equations, is developed. Almost perfect localization is predicted in sets of randomly distributed two-dimensional dipoles which are linearly coupled to the electric field of the incident wave. Striking qualitative similarities are observed between our results and effects found experimentally in microwave localization by random arrays of dielectric cylinders [R. Dalichaouch, J. P. Armstrong, S. Schultz, P. M. Platzman, and S. L. McCall, Nature (London) **354**, 53 (1991)].

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The localization of the electron wave functions originates from investigations of the electron transport in disordered solids, usually semiconductors [1]. In such media the propagation of electrons is altered by the presence of a random potential; this phenomenon is completely based on the interference effects in multiple elastic scattering. Since interference is a common property of all wave phenomena, many generalizations of the Anderson localization to other matter waves (neutrons) as well as classical waves (electromagnetic and acoustic waves) have been proposed [2,3] and, to some extent, also elaborated [4–6]. In this paper we focus our attention on electromagnetic waves. There is a variety of experimental investigations in this case, both in the optical and microwave domains. *Weak localization*, manifesting itself as enhanced coherent backscattering, is now experimentally established beyond any doubts. Striking coherent, back-directed peaks of intensity superimposed upon intensity of electromagnetic waves diffusely scattered from random media have been observed for different systems of randomly distributed scatterers forming effectively both two-dimensional [7,8] and three-dimensional [9–11] media. Weak localization is relatively well understood theoretically [12–14] and, as the coherent backscattering affects the diffusion constant describing the propagation of electromagnetic waves in strongly scattering random media, it is the precursor of *strong localization*. The question as to whether interference effects in strongly scattering random media can reduce the diffusion constant to zero producing purely localized states depends on the dimension of the sample under consideration. Despite some reasonable indications that strong localization could be possible in three-dimensional random dielectric structures (mainly some suspensions of TiO_2 spheres in air or in some low-refractive-index substances [8,15–18] have been considered) the convincing experimental demonstration has been given only for two dimensions [19]. In this case the strongly scattering medium has been provided by a set of dielectric cylinders randomly placed between two parallel aluminum plates on half the sites of a square lattice.

Despite the huge amount of existing literature, there still is a lack of sound theoretical models providing deeper insight into this interesting effect. To be realistic, such models should be based directly on the Maxwell equations. On the other hand, they should be simple enough to provide calculations without too many too-crude approximations. The main purpose of our paper is to construct such a model for the two-dimensional localization and to elaborate in detail its major consequences.

In the standard approach to localization of electromagnetic waves [2,20] a monochromatic wave,

$$\vec{E}(\vec{r},t)=\tfrac{1}{2}\{\vec{\mathscr{E}}(\vec{r})e^{-i\omega t}+\vec{\mathscr{E}}^{*}(\vec{r})e^{i\omega t}\}, \tag{1}$$

is called localized in a nondissipative dielectric medium if the squared modulus of the electric field $|\vec{\mathscr{E}}(\vec{r})|^2$ is localized. This definition is based on the analogy between the Helmholtz equation,

$$\{-\nabla^2+k_0^2[1-\epsilon(\vec{r})]\}\vec{\mathscr{E}}(\vec{r})=k_0^2\,\vec{\mathscr{E}}(\vec{r}), \tag{2}$$

and the time-independent Schrödinger equation. In the above formula $k_0=2\pi/\lambda_0$ and λ_0 denotes the wavelength in vacuum. It is known, however, that this analogy is far from complete. As follows from recent investigations, considerable care must be exercised in transforming results concerning localization of electrons to the case of electromagnetic waves [21,22]. To some extent the Helmholtz equation (2) may be interpreted as the eigenvalue equation for the wave function $\mathscr{E}(\vec{r})$ corresponding to the energy k_0^2. However, the presence of the energy-dependent "potential" $k_0^2[1-\epsilon(\vec{r})]$ strongly affects dynamical properties in scattering. For example, the transport velocity for multiple scattering processes can be very different from that observed for electrons [21,23]. These differences can be understood as a consequence of the fact that the counterpart of the Ward identity for electromagnetic waves contains additional "mass-enhancement" terms [21,22]. Closely related is the fact that the electric field vector cannot be interpreted as a probability amplitude. Because the conserved quantity is energy density of the field, we prefer to say that a monochromatic field (1) is localized if the time-averaged energy density of the field vanishes far from a certain region of space. It will be clear later that this approach is more natural and simplifies some considerations.

Since our discussion of localization is restricted to monochromatic fields only, we shall assume that the polarization of the dielectric medium providing localization is an oscillatory function of time $\vec{P}(\vec{r},t)=\mathrm{Re}[\vec{\mathscr{P}}(\vec{r})e^{-i\omega t}]$. Instead of

solving the Helmholtz equation (2) and then checking if the resulting wave obeys the transversality condition we prefer to study directly Maxwell equations in an integral form [24]

$$\vec{\mathscr{E}}(\vec{r})=\vec{\mathscr{E}}^{(0)}(\vec{r})+\vec{\nabla}\times\vec{\nabla}\times\int d^3r'\ \vec{\mathscr{P}}(\vec{r}\,')\frac{e^{ik_0|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}, \tag{3}$$

where $\vec{\mathscr{E}}^{(0)}(\vec{r})$ denotes a solution of the Maxwell equations in vacuum and $k_0=\omega/c$ is the wave number.

We believe that what really counts for localization is the scattering cross section and not the geometrical shape and real size of the scatterer. Therefore we will represent the dielectric cylinders located at the points $\vec{\rho}_a$ by two-dimensional dipoles

$$\vec{\mathscr{P}}(\vec{r})=\sum_{a=1}^{N}\vec{p}_a\,\delta^{(2)}(\vec{\rho}-\vec{\rho}_a). \tag{4}$$

Since the polarization of our system varies only at a certain plane, we have introduced cylindric coordinates $\vec{r}=(\vec{\rho},z)$ in the above formula. Let us stress that there is, in fact, no loss of generality. In practice any dielectric medium may be modeled by a set of discrete dipoles. This so-called coupled-dipole approximation was used to study light scattering by a dielectric sphere [25] and more recently to obtain the scattering coefficients of arbitrarily shaped particles [26]. There is, however, one important difference. The discrete dipole approximation works well only if there are many dipoles in a volume whose dimensions are of the order of the wavelength [24]. In numerical calculations performed on supercomputers, a single small dielectric particle is built out of about 10^6 dipoles [27]. In our case a single dielectric cylinder with diameter comparable to the wavelength [19] is modeled only by *one* dipole with properly adjusted scattering properties.

The crucial point is how each dipole should be coupled to the electromagnetic field. Of course the standard Lorenz-Lorentz formula [24] is now rather useless, because it is only approximately valid in the macroscopic limit. To provide a realistic and self-consistent description we must assume that the average energy is conserved in the scattering process. Therefore, if we isolate a single dipole, then the time-averaged field energy flux integrated over a closed surface Σ surrounding it should vanish for an arbitrary incident wave, namely,

$$\int_\Sigma d\vec{\sigma}\cdot\vec{\mathscr{S}}(\vec{r})=\frac{c}{4\pi}\frac{1}{2}\mathrm{Re}\int_\Sigma d\vec{\sigma}\cdot\{\vec{\mathscr{E}}(\vec{r})\times\vec{\mathscr{H}}^*(\vec{r})\}=0. \tag{5}$$

It is remarkable that this simple and obvious requirement gives an explicit form of the field-dipole coupling.

For the sake of simplicity let us now assume that both the free field and the medium are linearly polarized along the z axis $\vec{\mathscr{E}}^{(0)}(\vec{r})=\vec{e}_z\,\mathscr{E}^{(0)}(\vec{\rho})$, $\vec{\mathscr{P}}(\vec{r})=\vec{e}_z\,\mathscr{P}(\vec{\rho})$. It is now evident from Eq. (3) that the electric field of the wave radiated by the ath dipole reads as

$$\vec{\mathscr{E}}_a(\vec{r})=\vec{e}_z\,k_0^2\,p_a\,g^{(2)}(\vec{\rho}-\vec{\rho}_a), \tag{6}$$

where the Green function in two dimensions is expressed by the modified Bessel function of the second kind

$$g^{(2)}(\vec{\rho})=\int_{-\infty}^{+\infty}dz\,\frac{e^{ik_0|\vec{r}|}}{|\vec{r}|}=2\,K_0(-ik_0|\vec{\rho}|). \tag{7}$$

Therefore our discussion may be restricted to the scalar theory. This is impossible in three dimensions when the Green function is replaced by a tensor acting on the dipole moment and the proper description of the interaction between the field and the medium is more complicated [28].

Now we can perform the integration in Eq. (5) assuming that it is performed over a cylinder of unit height surrounding the ath dipole. The total energy flux may be split into three terms. The first term describes the total time-averaged energy flux integrated over a closed surface for a free field and thus vanishes. The second term corresponds to the time-averaged energy radiated by the ath dipole per unit time

$$\frac{c}{4\pi}\frac{1}{2}\mathrm{Re}\int_\Sigma d\vec{\sigma}\cdot\{\vec{\mathscr{E}}_a(\vec{r})\times\vec{\mathscr{H}}_a(\vec{r})\}=(2\pi)^2\,k_0^3\,|p_a|^2. \tag{8}$$

To calculate the last interference term let us use the following identity fulfilled by a field $\mathscr{E}(\vec{r})$ that obeys the free Maxwell equations inside a closed surface Σ:

$$\mathscr{E}(\vec{r})=\frac{1}{4\pi}\int_\Sigma d\vec{\sigma}\cdot\left\{\frac{e^{ik_0|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\vec{\nabla}\mathscr{E}(\vec{r}\,')-\mathscr{E}(\vec{r}\,')\vec{\nabla}\frac{e^{ik_0|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\right\}. \tag{9}$$

The above equation is known as the Kirchhoff integral formula [24]. After simple but lengthy calculations we finally arrive at the following conservation law which is equivalent to Eq. (5):

$$\pi k_0^2\,|p_a|^2=\mathrm{Im}\{p_a^*\,\mathscr{E}'(\vec{\rho}_a)\}, \tag{10}$$

where the field of the wave incident on the ath dipole,

$$\mathscr{E}'(\vec{\rho}_a)=\mathscr{E}^{(0)}(\vec{\rho}_a)+\sum_{b\neq a}\mathscr{E}_b(\vec{\rho}_a), \tag{11}$$

is the sum of the free field and the waves radiated by other dipoles.

Assuming that the dipole moment p_a is a linear function of the electric field $\mathscr{E}'(\vec{\rho}_a)$ we get from Eq. (10)

$$i\pi k_0^2\,p_a=\tfrac{1}{2}(e^{i\phi_a}-1)\mathscr{E}'(\vec{\rho}_a), \tag{12}$$

where ϕ_a is some arbitrary real number. Thus, to provide conservation of energy, the dipole moment must be coupled to the electric field of the incident wave by a complex "polarizability" $(e^{i\phi_a}-1)/2$. This fact is not specific for the considered two-dimensional case; it remains valid also in three dimensions [28]. We note that the field of the incident wave calculated at the dipole is finite as opposed to the total field which is not defined at the dipole. Inserting (12) into (6), using (11), and introducing the following convenient notation:

$$i\pi\,G_{ab}=\begin{cases}g^{(2)}(\vec{\rho}_a-\vec{\rho}_b) & \text{for } a\neq b\\ 0 & \text{for } a=b,\end{cases} \tag{13}$$

we finally arrive at the very simple set of linear algebraic equations

$$\vec{\mathscr{E}}'(\vec{\rho}_a)=\mathscr{E}^{(0)}(\vec{\rho}_a)+\tfrac{1}{2}\sum_{b=1}^{N}G_{ab}\,(e^{i\phi_b}-1)\,\mathscr{E}'(\vec{\rho}_b),\qquad a=1,\ldots,N, \tag{14}$$

determining the field acting on each dipole $\mathscr{E}'(\vec{\rho}_a)$ for a given free field $\mathscr{E}^{(0)}(\vec{\rho}_a)$. If we solve it and calculate the dipole moments we are able to find the electromagnetic field everywhere in space using the Maxwell equations (3).

To prove localization in the simple case of field sources vanishing outside of a certain region of space [like those given by (4)] one has to investigate the electromagnetic field

in the free space outside of the sources. According to our definition of localization, if an electromagnetic wave is localized then the time-averaged field energy density given by

$$\mathscr{W}(\vec{r})=(1/16\pi)\{|\vec{\mathscr{E}}(\vec{r})|^2+|\vec{\mathscr{H}}(\vec{r})|^2\} \tag{15}$$

must vanish far from the sources. It is evident from Eq. (3) that in this region the radiated field becomes zero. Therefore the free field must vanish on a sufficiently large closed surface. Thus the Kirchhoff integral formula implies that the free field is zero everywhere inside this surface. It is now evident that if the system of dipoles (4) provides localization of the electromagnetic wave then the system of equations (14) should have a nonzero solution for vanishing free field $\mathscr{E}^{(0)}=0$. This means that the eigenvalues ξ corresponding to the eigenvectors of the system (4)

$$\xi\,\mathscr{E}'(\vec{\rho}_a)=\mathscr{E}^{(0)}(\vec{\rho}_a), \quad a=1,\ldots,N, \tag{16}$$

describing localized waves should be equal to zero. In a general case the eigenvalues ξ depend on the positions of dipoles $\vec{\rho}_a$ and the phases ϕ_a describing their coupling to the field. However, assuming the same scattering properties of all dipoles, namely, $\phi_a=\phi$, we can express the eigenvalues as $\xi=1-[(e^{i\phi}-1)/2]\,\xi'$, where ξ' are the eigenvalues of the G matrix which depends only on the current positions of the dipoles. Note that in this case the eigenvectors of the system (14) are simultaneously eigenvectors of the G matrix. Thus, if $\vec{\rho}_a$ are given, then for each eigenvector of the system of equations (14) there exists a certain angle

$$\phi=\arg[1+(\xi'/2)]-\arg[\xi'/2], \tag{17}$$

for which the modulus of the corresponding eigenvalue ξ takes a minimal value $|\xi|_{\min}$ given by

$$\|1+(\xi'/2)|-|\xi'/2\|=|\xi|_{\min}\leq|\xi|\leq 1. \tag{18}$$

As a simple example let us consider a system of $N=100$ dipoles (4) distributed randomly in a square with the density n of one dipole per wavelength squared: $n/\lambda_0^{-2}=1.0$. We have calculated and diagonalized numerically the G matrix (13) describing this situation. Then we have chosen a certain eigenmode of the system (14) and checked if the corresponding eigenvalue can approach zero. According to Eq. (18) we have calculated the minimal value of its modulus and obtained $|\xi|_{\min}\propto 10^{-2}$. Therefore the field incident on each dipole $|\mathscr{E}'(\vec{\rho}_a)|^2$ is large compared to the free field $|\mathscr{E}^{(0)}(\vec{\rho}_a)|^2$ calculated at the dipole. Therefore the time-averaged energy density in the medium under consideration can be much greater than the energy density in the surrounding free space. Such a quasilocalization is practically indistinguishable from the perfect one. Of course the free field is not completely determined by specifying its values at the dipoles according to Eq. (16). However, we believe that it may be constructed in such a way that the time-averaged energy density of the free field will not exhibit local minima at the dipoles. To help those intuitive considerations, we have plotted in Fig. 1 the time-averaged energy density of the field corresponding to the considered eigenmode as a function of position. To avoid some infinities near the dipoles, this figure shows in fact a discrete function

$$\mathscr{W}'(\vec{\rho}_a)=(1/16\pi)\,\{|\vec{\mathscr{E}}'(\vec{\rho}_a)|^2+|\vec{\mathscr{H}}'(\vec{\rho}_a)|^2\}. \tag{19}$$

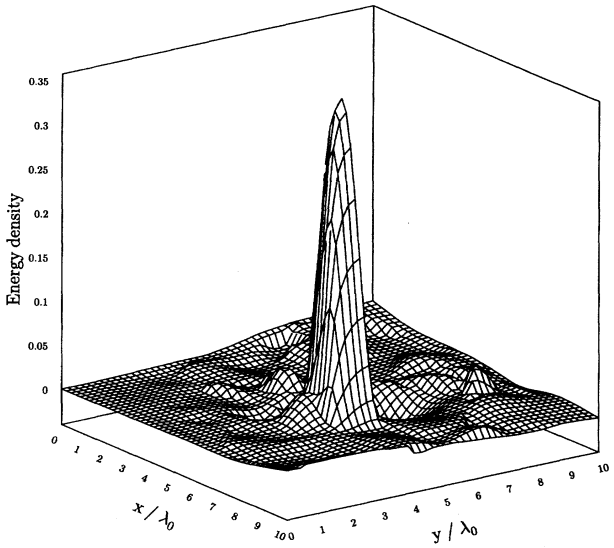

FIG. 1. Time-averaged energy density of the field localized in the medium consisting of 100 randomly distributed two-dimensional dipoles as a function of position. The density of dipoles is one dipole per wavelength squared.

Also, for simplicity, the free field has been set to zero, since, according to our assumption, a nonzero field may modify the plot only by 1%. In Fig. 2 we have a contour plot of the time-averaged energy density, corresponding to Fig. 1.

Obviously perfect localization (in the sense of our definition) is impossible in systems (4) consisting of a finite number of dipoles. However, for a fixed density of the medium under consideration quasilocalization becomes better for an increasing number of dipoles. To illustrate this statement we have plotted in Fig. 3 the minimal possible eigenvalue from *all* eigenmodes

$$|\Xi|_{\min}=\min_{j=1,\ldots,N}|\xi^{(j)}|_{\min}, \tag{20}$$

as a function of the number of dipoles N.

In fact, for a fixed system of dipoles given by Eq. (12), we see that Eq. (20), together with the equation determining the corresponding ϕ, Eq. (17), defines the dependence of the scattering properties of dipoles at the frequency of the localized field. Therefore we have constructed a model of a system of dielectric cylinders with the frequency-dependent per-

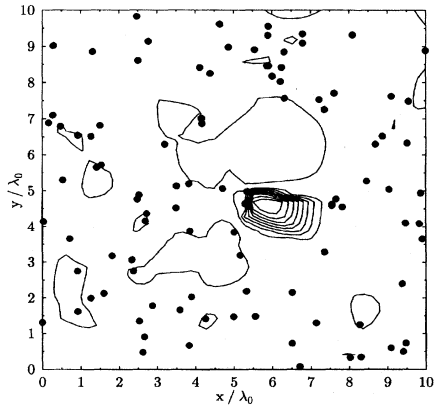

FIG. 2. Contour plot corresponding to Fig. 1. The positions of the dipoles are marked by black dots.

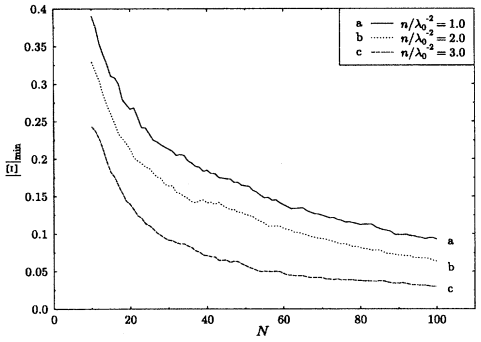

FIG. 3. $|\Xi|_{\min}$ as a function of the number of dipoles N for various densities n of the medium.

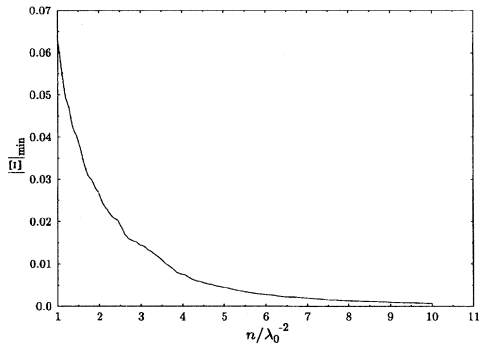

FIG. 4. $|\Xi|_{\min}$ as a function of the density of the medium n.

meability $\epsilon(\omega)$ chosen to provide the best possible localization for all frequencies. Indeed, as shown in Fig. 4, in such an abstract medium *quasi*localization takes place practically for all sufficiently large densities. Of course in an experiment the dielectric cylinders must have finite diameter to provide scattering cross section sufficiently large for localization which imposes a limit on the maximal density. Let us note that in Fig. 4 the sharp minima and maxima caused by resonances in a random medium and which are usually observed in experiments [19] are *not* present. This is of course due to our choice of the dependence of the scattering properties of the dipoles determined by ϕ as a function of frequency of localized electromagnetic waves.

In summary, we have presented an approach to localization of electromagnetic waves in two dimensions based on Maxwell equations in integral form. A dielectric medium providing localization is modeled by a system of discrete dipoles. Each dipole corresponds to a single dielectric cylinder. In this treatment the scattering properties of the medium may be eliminated from the considerations and the best possible localization for a given distribution of particles can be studied. It was shown that, in a medium consisting of a finite number of dielectric cylinders with specially chosen frequency-dependent permeability, a perfect photonic band gap is impossible. However, quasilocalization becomes better with increasing density of the medium or number of scattering particles. Instead of studying elastic scattering of electromagnetic waves by a pointlike particle using the t-matrix formalism and several crude approximations [29], we presented a simple method based on the Kirchhoff theorem. However, our general formula may also describe several unphysical systems. To solve this problem, a detailed analysis of the internal structure of the scatterer is needed. The generalization of the presented concepts to the three-dimensional vector case is a subject of a forthcoming paper [28]. We plan to analyze transport properties of our abstract medium, such as the mean free path, and check several heuristic criteria of localization.

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